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TRAFFIC FLOW MODELS WITH PHASE TRANSITIONS ON ROAD NETWORKS

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Abstract. The paper presents a review of the main analytical results available on the traffic flow model with phase transitions described in [10]. We also introduce a forthcoming existence result on road networks [14].

1. **Introduction.** The interest in traffic dynamics has considerably increased in the last decades. The modelling of pedestrian and vehicular traffic can be developed under different approaches. We can distinguish between microscopic (particle-based), mesoscopic (gas-kinetic) and macroscopic (fluid-dynamic) models. We refer the reader to the review paper [29] for an overview on the possible approaches, and the analysis and interpretation of various interesting phenomena occurring in traffic. A survey of the available mathematical models is given in [5, 23, 35].

The fluid-dynamic approach considers the evolution of macroscopic variables, such as the density of vehicles and their average velocity. Historically, one of the first continuous models introduced to describe traffic flow is the well known Lighthill-Whitham [38] and Richards [40] (LWR) model, which reads

$$\partial_t \rho + \partial_x [\rho v(\rho)] = 0, \tag{1}$$

where $\rho \in [0, R]$ is the mean traffic density, and $v(\rho)$, the mean traffic velocity, is a given non-negative non-increasing function. The maximal density $R > 0$ corresponds to a traffic jam. This scalar model expresses conservation of the number of cars, and relies on the assumption that the car speed depends only on the density (more complex closure relations between speed and density, involving the density gradient, can be assumed, see [5] and references therein). This phenomenological relation is valid in steady state conditions, and is not realistic in more complicated situations. In particular, as shown in Figure 1, the corresponding fundamental diagram in the $(\rho, \rho v)$-plane does not qualitatively match experimental data at high densities.

Later on, several second order models, i.e. models with two equations, were considered, see [2, 27, 39, 41, 42]. A third order model was presented in [28].

The diagram showed in Figure 1 suggests that a good traffic flow model should exhibit two qualitative different behaviors:

- for low densities, the flow is *free* and essentially analogous to that of the LWR model;

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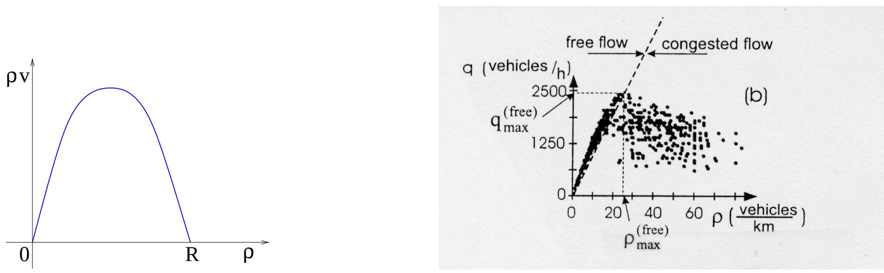

Figure 1. Left: standard flow for the LWR model. Right: experimental data, taken from [34]; here q denotes the flux ρv.

- at high densities the flow is *congested* and covers a 2-dimensional domain in the fundamental diagram; a "second order" model seems more appropriate to describe this dynamic.

Traffic flow models with phase transitions have been considered in the literature since the 60-ties (see Helbing [29, Section II] for a description of the features recovered by a detailed analysis of the fundamental diagram). In particular, we refer the reader to the scalar model by Drake, Schofer and May [19]. Another model has been introduced more recently by the author [24].

In the present paper we concentrate on the second order model with phase transitions introduced by Colombo [10]. This model has been conceived in order to reproduce the empirical flow-density relations showed in Figure 1 (right). From the analytical point of view, the model is well posed in the space of functions with bounded total variation. More precisely, when considering the Cauchy or the Initial-Boundary value problem, it is possible to construct a Lipschitz continuous semigroup of solutions, which is defined on a domain of functions with bounded total variation, see [15].

Here we are interested in extending the theory to road networks. The results available for networks concern the LWR model or the Aw-Rascle model, see [4, 7, 8, 20, 21, 22, 30, 31, 32]. No result was available for models with phase transitions up to now.

In [14] authors prove the existence of weak solutions on the whole network for initial data of bounded variation under the (technical) assumption that traffic keeps away from the zero velocity. The construction is based on the wave-front tracking method: We first consider Riemann problems at nodes, which are Cauchy problems with constant initial data on each road converging to a given junction. Notice that the only conservation of cars is not sufficient to determine a unique solution. Thus one has to prescribe solutions for every initial data and we call the relative mapping a *Riemann solver* at nodes. In the case studied here, we consider two Riemann solvers that are defined by generalizing to the phase transition model two Riemann solvers previously presented for the LWR model: The Riemann solver $\mathcal{R}^1_J$ was proposed for vehicular traffic in [8], while the Riemann solver $\mathcal{R}^2_J$ was introduced for telecommunication networks in [18]. The first prescribes a fixed distribution of traffic in outgoing roads, and then the maximization of the flux through the junction. The second maximizes the flux through the junction and then prescribes a distribution of traffic.

In the case the Riemann solver $\mathcal{R}^1_J$ is considered, we are able to construct piecewise constant approximations via wave-front tracking algorithm (see [6] for the general theory and [23] in the case of networks), using classical self-similar entropic solutions for Riemann problems inside roads and an assigned Riemann solver at junctions. To pass to the limit we rely on an estimate on the total variation of the flux.

The paper is organized as follows. Section 2 is devoted to the description of the model, and the classical Riemann solver. Section 3 collects the results on the well posedness of the Cauchy and the Initial-Boundary Value problem. Road networks are introduced in Section 4, where we describe the sets of attainable values at junctions, and the Riemann solvers at junctions are described in Section 5.

2. **Description of the model.** We consider the model introduced in [10]. It consists of a scalar LWR model coupled with the 2×2 system presented in [9]. The former applies to the states of *free* flow, while the latter to the *congested* states. A *phase transition* is a discontinuity separating a state of free traffic from one in the congested phase. More precisely, the model in [10] reads

$$\begin{array}{ll} \text{Free flow: } (\rho, q) \in \Omega_f & \text{Congested flow: } (\rho, q) \in \Omega_c \\ \partial_t \rho + \partial_x [\rho \cdot v] = 0 & \partial_t \rho + \partial_x [\rho \cdot v] = 0 \\ q = \rho V & \partial_t q + \partial_x [(q - Q) \cdot v] = 0 \\ v = v_f(\rho) = \left(1 - \frac{\rho}{R}\right) V & v = v_c(\rho, q) = \left(1 - \frac{\rho}{R}\right) \frac{q}{\rho} \,. \end{array} \tag{2}$$

Here, R is the maximal traffic density, V is the maximal traffic speed and Q is a parameter of the road under consideration related to the phenomenon of *wide jams*, see [10, 33]. The weighted linear momentum q is originally motivated by gas dynamics. It approximates the real flux ρv for ρ small compared to R.

It is assumed that if the initial data are entirely in the free (resp. congested) phase, then the solution will remain in the free (resp. congested) phase for all times. Thus we take Ω_f and Ω_c to be invariant sets for the corresponding equations. The resulting domain is given by $\Omega = \Omega_f \cup \Omega_c$, where

$$\begin{aligned} \Omega_f &= \left\{(\rho, q) \in [0, R] \times [0, +\infty[\,:\, v_f(\rho) \geq V_f ,\, q = \rho \cdot V \right\} , \\ \Omega_c &= \left\{(\rho, q) \in [0, R] \times [0, +\infty[\,:\, v_c(\rho, q) \leq V_c ,\, \tfrac{q - Q}{\rho} \in \left[\tfrac{Q^- - Q}{R}, \tfrac{Q^+ - Q}{R}\right]\right\} . \end{aligned}$$

Here, V_f and V_c are the threshold speeds, i.e. above V_f the flow is free, and below V_c the flow is congested. The parameters $Q^- \in]0, Q[$ and $Q^+ \in]Q, +\infty[$ depend on the environmental conditions and determine the width of the congested region.

Figure 2 shows that the shape of the invariant domain is in good agreement with experimental data. Notice that the sets are represented in the $(\rho, \rho v)$-plane.

Following [10, 15], throughout the present note we assume that the various parameters are strictly positive and satisfy

$$\begin{array}{l} V > V_f > V_c \,, \\ Q^+ \geq Q \geq Q^- \,, \end{array} \quad \frac{Q^+ - Q}{RV} < 1 \,, \; V_f = \frac{V - Q^+/R}{1 - (Q^+ - Q)/(RV)} \,. \tag{3}$$

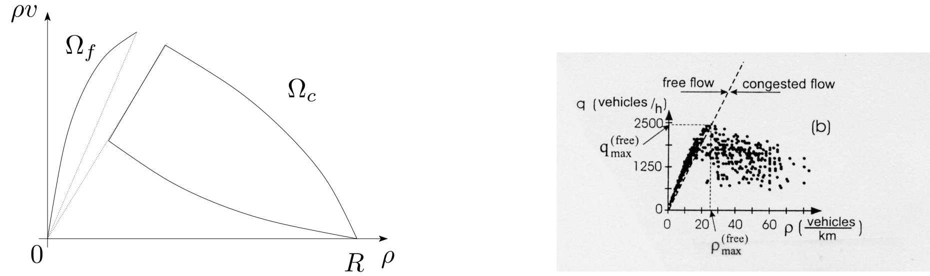

FIGURE 2. Left: invariant domain for (2). Right: experimental data, taken from [34]. The dotted straight lines exiting the origin are respectively $\rho v = \rho V_f$ and $\rho v = \rho V_c$. The continuous curves that border Ω_c are $\rho v = (1 - \rho/R)(Q + \rho(Q^{\pm} - Q)/R)$.

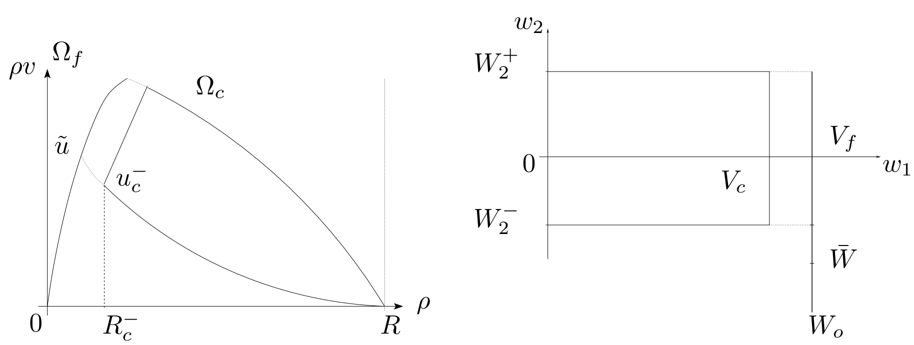

FIGURE 3. Notation used in the paper

We recall the basic informations on the 2×2 system on the right hand side of (2):

$$\begin{aligned}
&r_1(\rho, q) = \begin{bmatrix} \rho \\ q - Q \end{bmatrix}, && r_2(\rho, q) = \begin{bmatrix} R - \rho \\ \frac{R}{\rho} q \end{bmatrix}, \\
&\lambda_1(\rho, q) = \left(\frac{2}{R} - \frac{1}{\rho}\right) \cdot (Q - q) - \frac{Q}{R}, && \lambda_2(\rho, q) = v_c(\rho, q), \\
&\nabla \lambda_1 \cdot r_1 = 2\frac{Q - q}{R}, && \nabla \lambda_2 \cdot r_2 = 0, \\
&\mathcal{L}_1(\rho; \rho_o, q_o) = Q + \frac{q_o - Q}{\rho_o} \rho, && \mathcal{L}_2(\rho; \rho_o, q_o) = \frac{\rho}{\rho_o} \frac{R - \rho_o}{R - \rho} q_o, \\
&w_1 = v_c(\rho, q), && w_2 = \frac{q - Q}{\rho},
\end{aligned} \tag{4}$$

where r_i is the i-th right eigenvector, λ_i the corresponding eigenvalue and $\mathcal{L}_i$ is the i-Lax curve. In the Riemann coordinates (w_1, w_2), $\Omega_c = [0, V_c] \times [W_2^-, W_2^+]$. For $(\rho, q) \in \Omega_f$, we extend the corresponding Riemann coordinates (w_1, w_2) as follows. Let $\tilde{u} = (\tilde{\rho}, \tilde{\rho} V)$ be the point in Ω_f defined by $\tilde{\rho} = Q/(V - W_2^-)$. Define

$$w_1 = V_f \quad \text{and} \quad w_2 = \begin{cases} V - Q/\rho & \text{if } \rho \geq \tilde{\rho}, \\ v_f(\tilde{\rho}) - v_f(\rho) + V - Q/\tilde{\rho} & \text{if } \rho < \tilde{\rho}, \end{cases} \tag{5}$$

so that, in the Riemann coordinates, $\Omega_f = \{V_f\} \times [W_o, W_2^+]$, see Figure 3.

The 2×2 system describing the congested flow turns out to be hyperbolic, the second characteristic field is linearly degenerate but the first has an inflection point along the curve $q = Q$. Analogies between the solutions to (2) and real traffic features are given in [10].

For notational convenience, we introduce the following short form

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0 \tag{6}$$

for the model of phase transitions under consideration, with

$$\begin{cases} \mathbf{u} = (\rho, q) \quad \text{and} \quad \mathbf{f}(\mathbf{u}) = (\rho v_f(\rho), q v_f(\rho)), & \text{if} \quad (\rho, q) \in \Omega_f, \\ \mathbf{u} = (\rho, q) \quad \text{and} \quad \mathbf{f}(\mathbf{u}) = (\rho v_c(\rho, q), (q - Q) v_c(\rho, q)), & \text{if} \quad (\rho, q) \in \Omega_c. \end{cases}$$

2.1. **The Riemann problem.** We recall in this section the description of the classical Riemann solver for (2), i.e. the self-similar solution of the Cauchy problem

$$\begin{cases} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \\ \mathbf{u}_0(x) = \begin{cases} \mathbf{u}^l, & \text{if } x < 0, \\ \mathbf{u}^r, & \text{if } x > 0. \end{cases} \end{cases} \tag{7}$$

If the initial data $\mathbf{u}^l$, $\mathbf{u}^r$ are in the same phase, standard Lax solutions to the corresponding Riemann problem can be considered. Otherwise, following [10], admissible solutions are defined as follows.

Definition 2.1. If $\mathbf{u}^l \in \Omega_f$ and $\mathbf{u}^r \in \Omega_c$, then an **admissible solution** to (7) is a self-similar function $\mathbf{u} : \mathbb{R} \times [0, +\infty[\mapsto \Omega_f \cup \Omega_c$ such that, for some $\Lambda \in \mathbb{R}$, we have:

1. $\mathbf{u}(]-\infty, \Lambda t[) \subseteq \Omega_f$ and $\mathbf{u}(]\Lambda t, +\infty[) \subseteq \Omega_c$;
2. the functions

$$\mathbf{u}^-(x,t) = \begin{cases} \mathbf{u}(x,t) & \text{if } x < \Lambda t, \\ \mathbf{u}(\Lambda t-, t) & \text{if } x > \Lambda t, \end{cases} \tag{8}$$

$$\mathbf{u}^+(x,t) = \begin{cases} \mathbf{u}(\Lambda t+, t) & \text{if } x < \Lambda t, \\ \mathbf{u}(x,t) & \text{if } x > \Lambda t, \end{cases} \tag{9}$$

$$\tag{10}$$

   are Lax solutions to corresponding Riemann problems for (2) left, right, respectively;
3. the Rankine-Hugoniot condition

$$\rho(\Lambda t+, t)\, v_c\left(\mathbf{u}(\Lambda t+, t)\right) - \rho(\Lambda t-, t)\, v_f\left(\rho(\Lambda t+, t)\right) = \Lambda\left(\rho(\Lambda t+, t) - \rho(\Lambda t-, t)\right)$$

   holds for all $t > 0$.

If $\mathbf{u}^l \in \Omega_c$ and $\mathbf{u}^r \in \Omega_f$, the conditions are obtained by interchanging the roles of Ω_f, Ω_c and v_f, v_c.

Notice that condition 3 above ensures that the total number of car is conserved across phase transitions.

Definition 2.1 does not assure uniqueness. We are then led to introduce the notion of consistency [10].

Definition 2.2. Let $\mathcal{R} : (\mathbf{u}^l, \mathbf{u}^r) \mapsto \mathcal{R}(\mathbf{u}^l, \mathbf{u}^r)$ denote a Riemann solver, i.e. $x \mapsto \mathcal{R}(\mathbf{u}^l, \mathbf{u}^r)(x)$ is the solution of (7) computed at time $t = 1$. $\mathcal{R}$ is **consistent** if the following two conditions hold for all $\mathbf{u}^l,\ \mathbf{u}^m,\ \mathbf{u}^r \in \Omega_f \cup \Omega_c$, and $\bar{x} \in \mathbb{R}$:

$$\textbf{(C1)} \qquad \left.\begin{array}{ll} \mathcal{R}(\mathbf{u}^l, \mathbf{u}^m)(\bar{x}) & = \mathbf{u}^m \\ \mathcal{R}(\mathbf{u}^m, \mathbf{u}^r)(\bar{x}) & = \mathbf{u}^m \end{array}\right\} \quad \Rightarrow \quad \mathcal{R}(\mathbf{u}^l, \mathbf{u}^r) \ = \begin{cases} \mathcal{R}(\mathbf{u}^l, \mathbf{u}^m)\,, & \text{if } x < \bar{x}\,, \\ \mathcal{R}(\mathbf{u}^m, \mathbf{u}^r)\,, & \text{if } x \geq \bar{x}\,, \end{cases}$$

$$\textbf{(C2)} \qquad \mathcal{R}(\mathbf{u}^l, \mathbf{u}^r)(\bar{x}) \ = \mathbf{u}^m \quad \Rightarrow \quad \begin{cases} \mathcal{R}(\mathbf{u}^l, \mathbf{u}^m) = \begin{cases} \mathcal{R}(\mathbf{u}^l, \mathbf{u}^r)\,, & \text{if } x \leq \bar{x}\,, \\ \mathbf{u}^m\,, & \text{if } x > \bar{x}\,, \end{cases} \\ \mathcal{R}(\mathbf{u}^m, \mathbf{u}^r) = \begin{cases} \mathbf{u}^m\,, & \text{if } x < \bar{x}\,, \\ \mathcal{R}(\mathbf{u}^l, \mathbf{u}^r)\,, & \text{if } x \geq \bar{x}\,. \end{cases} \end{cases}$$

Essentially, **(C1)** states that whenever two solutions to two Riemann problems can be placed side by side, then their juxtaposition is again a solution to a Riemann problem. **(C2)** is the vice-versa.

We are now ready to construct the Riemann solver. We consider several different cases:

(A) The data in (7) are in the same phase, i.e. they are either both in Ω_f or both in Ω_c. Then the solution is the standard Lax solution to (2), left, resp. right, and no phase boundary is present.

(B) $\mathbf{u}^l \in \Omega_c$ and $\mathbf{u}^r \in \Omega_f$. We consider the points $\mathbf{u}^c \in \Omega_c$ and $\mathbf{u}^m \in \Omega_f$ implicitly defined by

$$\begin{aligned} \left(1 - \frac{\rho^c}{R}\right) \left(Q + w_2(\mathbf{u}^l)\rho^c\right) &= \rho^c V_c\,, \\ \left(1 - \frac{\rho^m}{R}\right) \left(Q + w_2(\mathbf{u}^l)\rho^m\right) &= \rho^m V \left(1 - \frac{\rho^m}{R}\right)\,. \end{aligned}$$

If $w_2(\mathbf{u}^l) > 0$, the solution is made of a 1-rarefaction from $\mathbf{u}^l$ to $\mathbf{u}^c$, a phase transition from $\mathbf{u}^c$ to $\mathbf{u}^m$ and a Lax wave from $\mathbf{u}^m$ to $\mathbf{u}^r$. If $w_2(\mathbf{u}^l) \leq 0$, we have a shock-like phase transition from $\mathbf{u}^l$ to $\mathbf{u}^m$ and a Lax wave from $\mathbf{u}^m$ to $\mathbf{u}^r$.

(C) $\mathbf{u}^l \in \Omega_f$ and $\mathbf{u}^r \in \Omega_c$ with $w_2(\mathbf{u}^l) \in [W_2^-, W_2^+]$. Consider the points $\mathbf{u}^c$ and $\mathbf{u}^m \in \Omega_c$ implicitly defined by

$$\begin{aligned} \left(1 - \frac{\rho^c}{R}\right) \left(Q + w_2(\mathbf{u}^l)\rho^c\right) &= \rho^c V_c\,, \\ \left(1 - \frac{\rho^m}{R}\right) \left(Q + w_2(\mathbf{u}^l)\rho^m\right) &= \rho^m w_1(\mathbf{u}^r)\,. \end{aligned}$$

If $w_2(\mathbf{u}^l) > 0$, the solution is made of a shock-like phase transition from $\mathbf{u}^l$ to $\mathbf{u}^m$ and a 2-contact discontinuity from $\mathbf{u}^m$ to $\mathbf{u}^r$. If $w_2(\mathbf{u}^l) \leq 0$, the solution displays a phase transition from $\mathbf{u}^l$ to $\mathbf{u}^c$, a 2-rarefaction from $\mathbf{u}^c$ to $\mathbf{u}^m$ and a 2-contact discontinuity from $\mathbf{u}^m$ to $\mathbf{u}^r$.

(D) $\mathbf{u}^l \in \Omega_f$ with $w_2(\mathbf{u}^l) < W_2^-$ and $\mathbf{u}^r \in \Omega_c$. Let $\mathbf{u}^m \in \Omega_c$ be the point on the lower boundary of Ω_c implicitly defined by

$$\left(1 - \frac{\rho^m}{R}\right) \left(Q + W_2^- \rho^m\right) = \rho^m w_1(\mathbf{u}^r)\,,$$

and consider the speed of the phase boundary joining $\mathbf{u}^l \in \Omega_f$ to $\mathbf{u}^m \in \Omega_c$

$$\Lambda(\mathbf{u}^l, \mathbf{u}^m) = \frac{\rho^l v_f(\rho^l) - \rho^m w_1(\mathbf{u}^r)}{\rho^l - \rho^m}.$$

Let $\mathbf{U}_c = (R_c, Q_c) \in \Omega_c$ be the point whose Riemann coordinates are (V_c, W_2^-). If $\lambda_1(\mathbf{U}_c) \geq \Lambda(\mathbf{u}^l, \mathbf{U}_c)$, the solution is a phase transition from $\mathbf{u}^l$ to $\mathbf{U}_c$, a

1-rarefaction from $\mathbf{U}_c$ to $\mathbf{u}^m$ and a 2-contact discontinuity from $\mathbf{u}^m$ to $\mathbf{u}^r$. Otherwise:

– If $\lambda_1(\mathbf{u}^m) \leq \Lambda(\mathbf{u}^l, \mathbf{u}^m)$, the solution is a phase transition from $\mathbf{u}^l$ to $\mathbf{u}^m$ followed by a 2-contact discontinuity from $\mathbf{u}^m$ to $\mathbf{u}^r$.
– If $\lambda_1(\mathbf{u}^m) > \Lambda(\mathbf{u}^l, \mathbf{u}^m)$, let $\mathbf{u}^c = (\rho^c, q^c) \in \Omega_c$ be implicitly defined by

$$\lambda_1(\mathbf{u}^c) = \Lambda(\mathbf{u}^l, \mathbf{u}^c)\,,$$

i.e. ρ^c is the bigger root of the equation

$$(Q - Q^-)\rho^2 - 2\rho^l(Q - Q^-)\rho + R^2(\rho^l v_f(\rho^l) - Q) + \rho^l R(2Q - Q^-) = 0$$

and $q^c = Q - \rho^c(Q - Q^-)/R$. Then the solution shows a phase transition from $\mathbf{u}^l$ to $\mathbf{u}^c$, an attached 1-rarefaction from $\mathbf{u}^c$ to $\mathbf{u}^m$ and a 2-contact discontinuity from $\mathbf{u}^m$ to $\mathbf{u}^r$.

3. **Well posedness.** In the literature, several results deal with the solution to Riemann problems in presence of phase transitions, see for instance [17, 36, 37]. Other works prove the global in time well posedness of the Cauchy problem, but with initial data that are perturbations of a given phase boundary, see for instance [11, 12]. On the contrary, the results presented in this section do not require *a priori* bounds on the number of phase boundaries that are present in the data and in the solution. From the analytical point of view, this is a first example of a system of conservation laws developing phase transitions whose well posedness is proved *globally*, i.e. for all initial data attaining values in a given set and with bounded total variation.

From the traffic point of view, well posedness allows to consider various control and optimization problems, see [16].

3.1. **The Cauchy Problem.** In this case, (6) is supplemented with a given value of the solution at time $t = 0$. More precisely, we assume that an initial datum $\mathbf{u}_0 \in \Omega$ is given and we set

$$\mathbf{u}(., t = 0) = \mathbf{u}_0. \tag{11}$$

We introduce the notations:

$$X \;=\; \mathbf{L^1}(\mathbb{R};\Omega)\,, \qquad \mathrm{TV}(\mathbf{u}) \;=\; \mathrm{TV}(\rho) + \mathrm{TV}(q)\,. \tag{12}$$

Definition 3.1. Fix $M > 0$ and X as above. A map $S\colon \mathbb{R}^+ \times \mathcal{D} \mapsto \mathcal{D}$ is an *M-Riemann Semigroup* (M-RS) if the following holds:

(RS1) $\mathcal{D} \supseteq \{\mathbf{u} \in X : \mathrm{TV}(\mathbf{u}) \leq M\}$;
(RS2) $S_0 = \mathrm{Id}$ and $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$;
(RS3) there exists an $L = L(M)$ such that for t_1, t_2 in $\mathbb{R}^+$ and $\mathbf{u}_1, \mathbf{u}_2$ in $\mathcal{D}$,

$$\|S_{t_1}\mathbf{u}_1 - S_{t_2}\mathbf{u}_2\|_{\mathbf{L^1}} \leq L \cdot \left(\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{L^1}} + |t_1 - t_2|\right);$$

(RS4) if $\mathbf{u} \in \mathcal{D}$ is piecewise constant, then for t small, $S_t\mathbf{u}$ coincides with the gluing of solutions to Riemann problems.

By *"solutions to Riemann problems"* we refer here to those defined in Section 2. Properties *(RS1)–(RS4)* provide the natural extension of [6, Definition 9.1] to the present case.

We are now ready to state the main result of this section, namely the existence of an M-RS generated by the Cauchy problem for (2).

Theorem 3.2. *For any positive M, the system* (2) *generates an M-RS $S\colon \mathbb{R}^+ \times \mathcal{D} \mapsto \mathcal{D}$. Moreover*

(CP1) *for all* $\mathbf{u}_0 \in \mathcal{D}$, *the orbit* $t \mapsto S_t\mathbf{u}_0$ *is a weak entropic solution to* (2) *with initial datum* $\mathbf{u}_0$*;*

(CP2) *any two* M*–RS coincide up to the domain;*

(CP3) *the solutions yielded by* S *can be characterized as viscosity solutions, in the sense of* [6, Theorem 9.2].

(CP4) $\mathcal{D} \subseteq \left\{\mathbf{u} \in X \colon \mathrm{TV}(\mathbf{u}) \leq \widehat{M}\right\}$ *for a positive* $\widehat{M}$ *depending only on* M*.*

The proof can be found in [15, § 4.2]. Observe that the description of several realistic situations requires suitable source terms in the right hand sides of model (2). The techniques in [3, 13] can then be applied.

3.2. **The Initial-Boundary Value Problem.** From the point of view of traffic flow, it is natural to consider Initial-Boundary Value problems (IBVP). We start considering the case of a road starting at $x = 0$ where the inflow $\tilde{f}(t)$ is regulated. This leads to study the following Riemann problem with boundary

$$\begin{array}{ll} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0 & t \geq 0\,, \quad x \geq 0 \\ \mathbf{u}(0,x) = \bar{\mathbf{u}} & x \geq 0\,, \\ (\rho v)(t,0) = \tilde{f} & t \geq 0\,. \end{array} \tag{13}$$

We denote the maximum possible flow along the considered road by $F = R_f V_f$.

When considering model (2), we assume that, besides (3), also

$$\left(1 - \frac{Q^+}{RV}\right) \cdot \left(\frac{Q^+}{Q} - 1\right) < 1 \tag{14}$$

holds. Condition (14) ensures that $\sup_{\Omega_f \cup \Omega_c} \lambda_1 < 0$, hence all waves of the first family are exiting the domain $x \geq 0$, $t \geq 0$ and the definition of solution introduced in [26], see also [1, Definition NC], applies: the boundary data $\tilde{f}$ is attained in the sense that

$$\lim_{x \to 0+} \rho(t,x) \cdot v(t,x) = \tilde{f} \qquad \text{for a.e. } t \geq 0\,.$$

Proposition 1. *With reference to problem* (13)*, if* (3), (14) *hold for* (2)*, then for all* $\bar{\mathbf{u}} \in \Omega_f \cup \Omega_c$*, there exists a threshold* $f^{\max} = f^{\max}(\bar{\mathbf{u}})$ *such that for all* $\tilde{f} \in [0, f^{\max}]$ *the Riemann problem for* (13) *admits a solution in the sense of* [1, Definition NC]. *More precisely, there exists a unique state* $\tilde{\mathbf{u}} \in \Omega_f \cup \Omega_c$ *such that the flow at* $\tilde{\mathbf{u}}$ *is* $\tilde{f}$ *and the solution to the standard Riemann problem for* (2) *with data* $u_L = \tilde{\mathbf{u}}$ *and* $u_R = \bar{\mathbf{u}}$ *consists only of waves having positive speed.*

1. *If* $\bar{\mathbf{u}} \in \Omega_f$*, then* $f^{\max} = F$ *and* $\tilde{\mathbf{u}}$ *is in* Ω_f*. The solution consists of a* 2*-wave in the free phase.*
2. *If* $\bar{\mathbf{u}} \in \Omega_c$*, then there exists a* $f^{\min} = f^{\min}(\bar{\mathbf{u}})$ *such that:*
   (a) *If* $f^{\min} \leq \tilde{f} \leq f^{\max}$*,* $\tilde{\mathbf{u}}$ *is the unique intersection between the curve* $\rho v(\mathbf{u}) = \tilde{f}$ *and the* 2*-wave through* $\bar{\mathbf{u}}$*. The solution consists of a simple* 2*-wave.*
   (b) *If* $\tilde{f} < f^{\min}$*, then* $\tilde{\mathbf{u}}$ *is the unique state in* Ω_f *such that* $\tilde{\rho} v_f(\tilde{\rho}) = \tilde{f}$*. The solution consists of a phase boundary and a* 2*-wave.*

The thresholds $f^{\min}$ and $f^{\max}$ are showed in Figure 4 and are given explicitly in [15], where the proof of Proposition 1 is detailed. Note that, as remarked in [15], the incoming flow $\tilde{f}$ can be slightly greater than the flow $\bar{\rho}\, v(\bar{\rho})$ present on the road. This is not allowed by the LWR model.

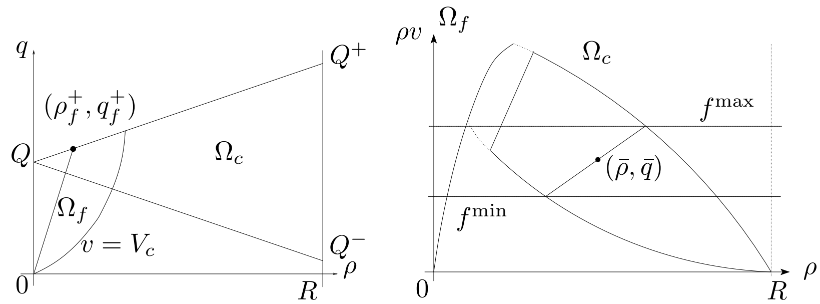

FIGURE 4. Notation used in Proposition 1

Once the Riemann solver is available, well posedness for the Initial-Boundary Value Problem can be proved as in [15], for all initial and boundary data with bounded total variation. Remark that due to the presence of phase boundaries, the number of waves entering the domain $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$ can not be *a priori* established.

We consider now the somewhat symmetric case of a road whose outflow at $x = 0$ is regulated. At the level of Riemann problem, this can be modeled by

$$\begin{array}{ll} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0 & t \geq 0\,, \quad x \leq 0 \\ \mathbf{u}(0,x) = \bar{\mathbf{u}} & x \leq 0\,, \\ (\rho v)(t,0) \leq \tilde f & t \geq 0\,. \end{array} \tag{15}$$

Proposition 2. *With reference to (15), conditions* (3) *and* (14) *imply that for all* $\bar{\mathbf{u}} \in \Omega_f \cup \Omega_c$*, and for all possible flows* $\tilde f \in [0, F]$ *the Riemann problem (15) admits a solution in the sense of* [1, Definition NC]*. More precisely, there exists a unique state* $\tilde{\mathbf{u}} \in \Omega_f \cup \Omega_c$ *such that the flow at* $\tilde{\mathbf{u}}$ *is less than or equal to* $\tilde f$ *and the standard solution to the Riemann problem for model* (2) *with data* $u_L = \bar{\mathbf{u}}$ *and* $u_R = \tilde{\mathbf{u}}$ *consists only of waves having negative speed. If the flow at* $\bar{\mathbf{u}}$ *is less than or equal to* $\tilde f$*, then* $\tilde{\mathbf{u}} = \bar{\mathbf{u}}$*, otherwise:*

1. *If* $\bar{\mathbf{u}} \in \Omega_f$*, then* $\tilde{\mathbf{u}}$ *is in* Ω_c*. The solution consists of a phase transition possibly followed by a* 1*-wave in the congested phase.*
2. *If* $\bar{\mathbf{u}} \in \Omega_c$*, then the solution consists of a simple* 1*-wave,* $\tilde{\mathbf{u}}$ *being the intersection of the* 1*-Lax curve through* $\bar{\mathbf{u}}$ *and the line* $\rho v = \tilde f$*.*

A proof is given in [25].

4. **Road network with phase transitions.** A road network is a couple $(\mathcal{I}, \mathcal{J})$, where $\mathcal{I}$ is a finite collection of unidirectional roads and $\mathcal{J}$ is the set of junctions. Each road is modelled by real intervals I_i, $i = 1, \ldots, N$, while each junction J consists of two sets $Inc(J) \subset \{1, \ldots, N\}$ and $Out(J) \subset \{1, \ldots, N\}$ corresponding to incoming and outgoing roads of J.

Given a junction J, a Riemann problem at J is a Cauchy problem with initial data constant on each incoming and outgoing road.

As for classical Riemann problems on a real line, we look for self-similar, centered solutions, which are the building blocks to construct solutions to Cauchy problems. We are now ready to introduce the key concept of Riemann solver at J.

Definition 4.1. Consider a junction J and assume for simplicity $Inc(J) = \{1, \dots, n\}$, $Out(J) = \{n+1, \dots, n+m\}$. A Riemann solver $\mathcal{R}_J$ is a function

$$\begin{array}{cccc} \mathcal{R}_J: & (\Omega_f \cup \Omega_c)^{n+m} & \longrightarrow & (\Omega_f \cup \Omega_c)^{n+m} \\ & (\mathbf{u}_{1,0}, \dots, \mathbf{u}_{n+m,0}) & \longmapsto & (\hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_{n+m}) \end{array}$$

satisfying the following

1. $\sum_{i=1}^{n} \mathbf{f}_1(\hat{\mathbf{u}}_i) = \sum_{j=n+1}^{n+m} \mathbf{f}_1(\hat{\mathbf{u}}_j)$, where $\mathbf{f}_1$ is the first component of $\mathbf{f}$;
2. for every $i \in \{1, \dots, n\}$, the classical Riemann problem
$$\begin{cases} \mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, & x \in \mathbb{R},\ t > 0, \\ \mathbf{u}(0,x) = \begin{cases} \mathbf{u}_{i,0}, & \text{if } x < 0, \\ \hat{\mathbf{u}}_i, & \text{if } x > 0, \end{cases} \end{cases}$$
is solved by waves with negative speed;
3. for every $j \in \{n+1, \dots, n+m\}$, the classical Riemann problem
$$\begin{cases} \mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, & x \in \mathbb{R},\ t > 0, \\ \mathbf{u}(0,x) = \begin{cases} \hat{\mathbf{u}}_j, & \text{if } x < 0, \\ \mathbf{u}_{j,0}, & \text{if } x > 0, \end{cases} \end{cases}$$
is solved by waves with positive speed.

To effectively describe a solution to Riemann problems at J, a Riemann solver needs to satisfy the following consistency condition:

Definition 4.2. We say that a Riemann solver $\mathcal{R}_J$ satisfies the consistency condition if

$$\mathcal{R}_J(\mathcal{R}_J(\mathbf{u}_{1,0}, \dots, \mathbf{u}_{n+m,0})) = \mathcal{R}_J(\mathbf{u}_{1,0}, \dots, \mathbf{u}_{n+m,0})$$

for every $(\mathbf{u}_{1,0}, \dots, \mathbf{u}_{n+m,0}) \in (\Omega_f \cup \Omega_c)^{n+m}$.

In what follows we will assume (14), in order to have in the congested phase 1-waves always moving with negative speed.

4.1. **Incoming roads: attainable values at the junction.** To respect condition 2 of Definition 4.1 only waves with negative speed can be produced on incoming roads. Thus we determine all states which can be connected to an initial state (to the right) by waves with negative speed. In particular, we determine the maximum flux $\gamma_i^{\max}$ that can be reached from an initial datum $\mathbf{u}_{i,0} = (\rho_{i,0}, q_{i,0})$ by means of waves with negative speed only.

We start describing the sets of fluxes corresponding to states that can be connected to $\mathbf{u}_{i,0}$ on the right using non positive waves only. We use the notations introduced in Section 2.1, Cases **(B)-(D)**, where we set $\mathbf{u}_{i,0} = \mathbf{u}^l$. Moreover, we introduce the velocities V_1 and V_2 defined as follows:

- $V_1 := v_f(\rho_1)$, where $\rho_1 \in \Omega_f$ is the smaller root of the equation $\rho_1 v_f(\rho_1) = R_c V_c$;
- $V_2 := v_f(\rho_2)$, where $\rho_2 \in \Omega_f$ is the smaller root of the equation
$$\left(1 - \frac{\rho_2}{R}\right)\left(Q + \frac{Q_- - Q}{R}\rho_2\right) = \rho_2 V \left(1 - \frac{\rho_2}{R}\right).$$

We refer the reader to Figure 5 for help in understanding notations.

The sets of reachable fluxes are then given by

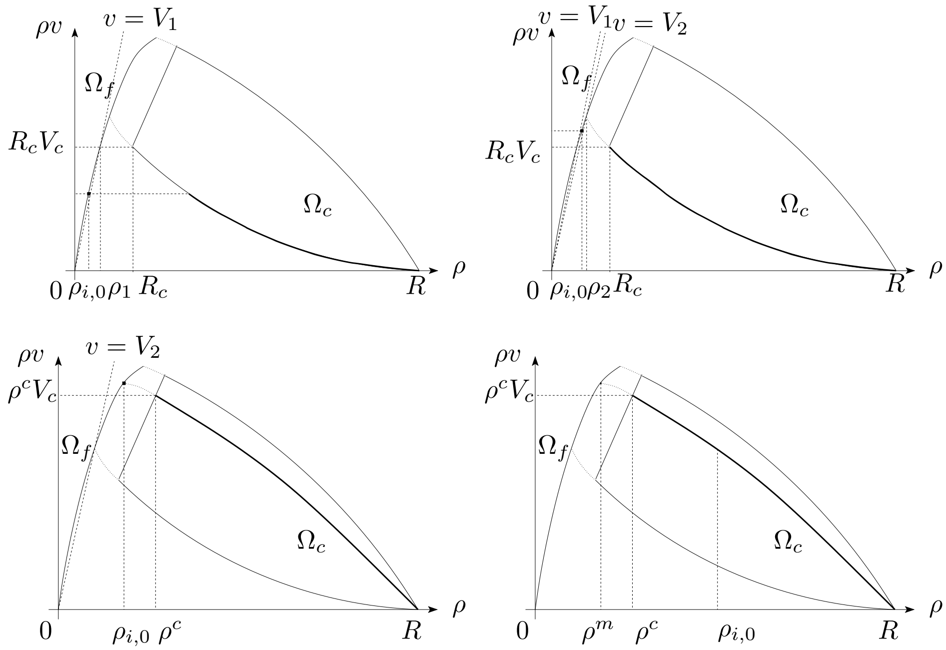

FIGURE 5. Notations used in the definition of $\mathcal{O}_i$, $i = 1, 2$.

$$\mathcal{O}_i = \begin{cases} [0, \rho_{i,0} v_f(\rho_{i,0})] & \text{if } \mathbf{u}_{i,0} \in \Omega_f, v_f(\rho_{i,0}) \geq V_1 \,, \\ [0, R_c V_c] \cup \{\rho_{i,0} v_f(\rho_{i,0})\} & \text{if } \mathbf{u}_{i,0} \in \Omega_f, V_2 \leq v_f(\rho_{i,0}) \leq V_1 (\text{Case } \mathbf{(D)}, \text{ Sec. } 2.1) \,, \\ [0, \rho^c V_c] \cup \{\rho_{i,0} v_f(\rho_{i,0})\} & \text{if } \mathbf{u}_{i,0} \in \Omega_f, v_f(\rho_{i,0}) \leq V_2 (\text{Case } \mathbf{(C)}, \text{ Sec. } 2.1) \,, \\ [0, \rho^c V_c] \cup \{\rho^m v_f(\rho^m)\} & \text{if } \mathbf{u}_{i,0} \in \Omega_c (\text{Case } \mathbf{(B)}, \text{ Sec. } 2.1) \,, \end{cases} \tag{16}$$

for $i = 1, \ldots, n$. We observe that the sets $\mathcal{O}_i$ are non convex. In order to have continuous dependence of solutions, we have to get convexity removing the *metastable* states from the attainable sets. This choice is consistent with the idea that such states should appear in a transient situation, which should not happen at a junction. Hence we define the corresponding maximum fluxes as follows:

$$\gamma_i^{\max} = \begin{cases} \rho_{i,0} v_f(\rho_{i,0}) & \text{if } \mathbf{u}_{i,0} \in \Omega_f, v_f(\rho_{i,0}) \geq V_1 \,, \\ R_c V_c & \text{if } \mathbf{u}_{i,0} \in \Omega_f, V_2 \leq v_f(\rho_{i,0}) \leq V_1 (\text{Case } \mathbf{(D)}, \text{ Sec. } 2.1) \,, \\ \rho^c V_c & \text{if } \mathbf{u}_{i,0} \in \Omega_f, v_f(\rho_{i,0}) \leq V_2 (\text{Case } \mathbf{(C)}, \text{ Sec. } 2.1) \,, \\ \rho^c V_c & \text{if } \mathbf{u}_{i,0} \in \Omega_c (\text{Case } \mathbf{(B)}, \text{ Sec. } 2.1) \,. \end{cases} \tag{17}$$

Proposition 3. *Given an initial datum $\mathbf{u}_{i,0}$ on an incoming road and $\hat{\gamma} \in [0, \gamma_i^{max}]$, there exists a unique $\hat{\mathbf{u}}_i$ such that the Riemann problem $(\mathbf{u}_{i,0} \hat{\mathbf{u}}_i)$ is solved by waves with negative speed and $\mathbf{f}_1(\hat{\mathbf{u}}_i) = \hat{\gamma}$.*

4.2. **Outgoing roads: maximal flux at the junction.** To respect condition 3 of Definition 4.1 only waves with positive speed can be produced on outgoing roads. Thus we determine all states, and the corresponding set of fluxes, which can be connected to an initial state $\mathbf{u}_{j,0}$ (to the left) using waves with positive speed.
We introduce the fluxes F and $f^{\max}$ defined as follows (see Figure 6):

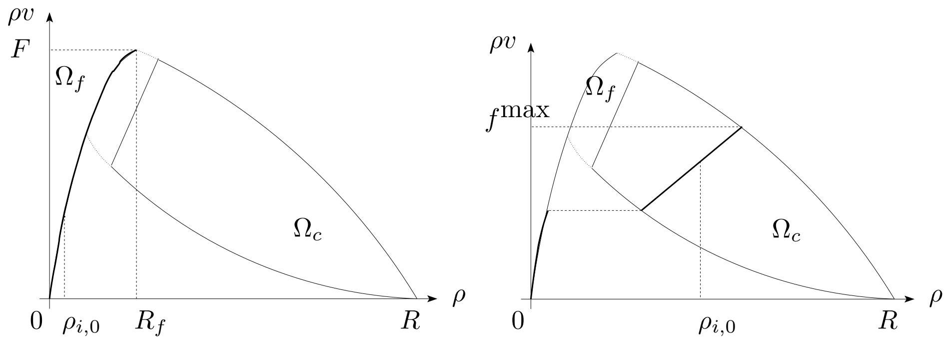

FIGURE 6. Notations used in the definition of $\mathcal{O}_j$, $j = 3, 4$.

- $F = R_f v_f(R_f) = \max_{\rho\in\Omega_f} \rho v_f(\rho) > \max_{(\rho,q)\in\Omega_c} \rho v_c(\rho, q)$ is the maximal flux supported by the road;
- for $\mathbf{u}_{j,0} \in \Omega_c$, $f^{\max} = f^{\max}(\mathbf{u}_{j,0}) = \rho^{\max} v_c(\rho^{\max}, q^{\max})$, where $\rho^{\max}$ is the bigger root of the equation

$$\left(1 - \frac{\rho^{\max}}{R}\right)\left(Q + \frac{Q_+ - Q}{R}\rho^{\max}\right) = \rho^{\max} v_c(\rho_{j,0}, q_{j,0}),$$

  and $q^{\max} = Q + \rho^{\max}(Q_+ - Q)/R$.

The sets of reachable fluxes are given by

$$\mathcal{O}_j = \begin{cases} [0, F] & \text{if } \mathbf{u}_{j,0} \in \Omega_f, \\ [0, f^{\max}] & \text{if } \mathbf{u}_{j,0} \in \Omega_c, \end{cases} \tag{18}$$

for $j = n+1, \ldots, n+m$. Since the sets $\mathcal{O}_j$ are convex, the corresponding maximum fluxes are defined accordingly:

$$\gamma_j^{\max} = \begin{cases} F & \text{if } \mathbf{u}_{j,0} \in \Omega_f, \\ f^{\max} & \text{if } \mathbf{u}_{j,0} \in \Omega_c. \end{cases} \tag{19}$$

Proposition 4. *Given an initial datum $\mathbf{u}_{j,0}$ on an outgoing road and $\hat{\gamma} \in [0, \gamma_j^{max}]$, there exists a unique $\hat{\mathbf{u}}_j \in \mathcal{O}_j$ such that the Riemann problem $(\hat{\mathbf{u}}_j, \mathbf{u}_{j,0})$ is solved by waves with positive speed and $\mathbf{f}_1(\hat{\mathbf{u}}_j) = \hat{\gamma}$.*

5. **Riemann solvers at junctions.** In this section we describe two Riemann solvers introduced in [8, 18] for a LWR model, which can be used also for the phase transition model (2).

5.1. **Riemann Solver $\mathcal{R}_J^1$.** We define a Riemann solver similar to that introduced in [8] for vehicular traffic.

First, we need to define a suitable set of matrices. Consider the set

$$\mathcal{A} := \left\{ A = \{a_{ji}\}_{i=1,\ldots,n,\ j=n+1,\ldots,n+m} : \begin{array}{c} 0 < a_{ji} < 1\ \forall i, j, \\ \sum_{j=n+1}^{n+m} a_{ji} = 1\ \forall i \end{array} \right\}. \tag{20}$$

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. For every $i = 1, \ldots, n$, we denote $H_i = \{e_i\}^\perp$. If $A \in \mathcal{A}$, then we write, for every $j = n+1, \ldots, n+m$, $a_j =$

$(a_{j1}, \ldots, a_{jn}) \in \mathbb{R}^n$ and $H_j = \{a_j\}^\perp$. Let $\mathcal{K}$ be the set of indices $\mathbf{k} = (k_1, ..., k_\ell)$, $1 \le \ell \le n-1$, such that $0 \le k_1 < k_2 < \cdots < k_\ell \le n+m$ and for every $\mathbf{k} \in \mathcal{K}$ define

$$H_{\mathbf{k}} = \bigcap_{h=1}^{\ell} H_{k_h}.$$

Writing $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$ and following [8] we define the set

$$\mathfrak{N} := \left\{ A \in \mathcal{A} : \mathbf{1} \notin H_{\mathbf{k}}^\perp \text{ for every } \mathbf{k} \in \mathcal{K} \right\}. \tag{21}$$

Notice that, if $n > m$, then $\mathfrak{N} = \emptyset$. The matrices of $\mathfrak{N}$ will give rise to a unique solution to Riemann problems at J.

1. Fix a matrix $A \in \mathfrak{N}$ and consider the closed, convex and not empty set

$$\Lambda = \left\{ (\gamma_1, \cdots, \gamma_n) \in \prod_{i=1}^{n} [0, \gamma_i^{\max}] : A \cdot (\gamma_1, \cdots, \gamma_n)^T \in \prod_{j=n+1}^{n+m} [0, \gamma_j^{\max}] \right\}. \tag{22}$$

2. Find the point $(\bar\gamma_1, \ldots, \bar\gamma_n) \in \Lambda$ which maximizes the function

$$E(\gamma_1, \ldots, \gamma_n) = \gamma_1 + \cdots + \gamma_n, \tag{23}$$

and define $(\bar\gamma_{n+1}, \ldots, \bar\gamma_{n+m})^T := A \cdot (\bar\gamma_1, \ldots, \bar\gamma_n)^T$. Since $A \in \mathfrak{N}$, the point $(\bar\gamma_1, \ldots, \bar\gamma_n)$ is uniquely defined.

3. For every $i \in \{1, \ldots, n\}$, set $\hat{\mathbf{u}}_i$ either to $\mathbf{u}_{i,0}$ if $\mathbf{f}_1(\mathbf{u}_{i,0}) = \bar\gamma_i$, or to the solution to $\mathbf{f}_1(\mathbf{u}) = \bar\gamma_i$ given by Proposition 3. For every $j \in \{n+1, \ldots, n+m\}$, set $\hat{\mathbf{u}}_j$ either to $\mathbf{u}_{j,0}$ if $\mathbf{f}_1(\mathbf{u}_{j,0}) = \bar\gamma_j$, or to the solution to $\mathbf{f}_1(\mathbf{u}) = \bar\gamma_j$ given by Proposition 4. Finally, set

$$\mathcal{R}_J^1(\mathbf{u}_{1,0}, \ldots, \mathbf{u}_{n+m,0}) = (\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_{n+m}). \tag{24}$$

It is easy to verify that $\mathcal{R}_J^1$ satisfies the consistency condition (CC).

5.2. **Riemann Solver $\mathcal{R}_J^2$.** In this subsection we define a Riemann solver similar to that introduced in [18].

First let us define

$$\Theta = \left\{ \theta = (\theta_1, \ldots, \theta_{n+m}) \in \mathbb{R}^{n+m} : \begin{array}{c} \theta_1 > 0, \cdots, \theta_{n+m} > 0, \\ \sum_{i=1}^n \theta_i = \sum_{j=n+1}^{n+m} \theta_j = 1 \end{array} \right\}. \tag{25}$$

1. Fix $\theta \in \Theta$ and define

$$\Gamma_{inc} = \sum_{i=1}^{n} \gamma_i^{\max}, \quad \Gamma_{out} = \sum_{j=n+1}^{n+m} \gamma_j^{\max},$$

then the maximal possible through-flow at the crossing is

$$\Gamma = \min \{\Gamma_{inc}, \Gamma_{out}\}.$$

2. Introduce the closed, convex and not empty sets

$$\begin{aligned} \mathbf{Inc} &= \left\{ (\gamma_1, \ldots, \gamma_n) \in \prod_{i=1}^{n} [0, \gamma_i^{\max}] : \sum_{i=1}^{n} \gamma_i = \Gamma \right\} \\ \mathbf{Out} &= \left\{ (\gamma_{n+1}, \ldots, \gamma_{n+m}) \in \prod_{j=n+1}^{n+m} [0, \gamma_j^{\max}] : \sum_{j=n+1}^{n+m} \gamma_j = \Gamma \right\}. \end{aligned}$$

3. Denote with $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$ the orthogonal projection on the convex set **Inc** of the point $(\Gamma\theta_1, \ldots, \Gamma\theta_n)$ and with $(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m})$ the orthogonal projection on the convex set **Out** of the point $(\Gamma\theta_{n+1}, \ldots, \Gamma\theta_{n+m})$.
4. For every $i \in \{1, \ldots, n\}$, set $\hat{\mathbf{u}}_i$ either to $\mathbf{u}_{i,0}$ if $\mathbf{f}_1(\mathbf{u}_{i,0}) = \bar{\gamma}_i$, or to the solution to $\mathbf{f}_1(\mathbf{u}) = \bar{\gamma}_i$ given by Proposition 3. For every $j \in \{n+1, \ldots, n+m\}$, set $\hat{\mathbf{u}}_j$ either to $\mathbf{u}_{j,0}$ if $\mathbf{f}_1(\mathbf{u}_{j,0}) = \bar{\gamma}_j$, or to the solution to $\mathbf{f}_1(\mathbf{u}) = \bar{\gamma}_j$ given by Proposition 4. Finally, set

$$\mathcal{R}_J^2(\mathbf{u}_{1,0}, \ldots, \mathbf{u}_{n+m,0}) = (\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_{n+m}) \, . \tag{26}$$

Also in this case it is easy to verify that $\mathcal{R}_J^2$ satisfies the consistency condition (CC).

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